

V. *A Letter from Mr. Colin Mac Laurin, Math. Prof. Edinburg. F. R. S. to Mr. John Machin, Astr. Prof. Gresh. & Secr. R. S. concerning the Description of Curve Lines. Communicated to the Royal Society on December 21, 1732.*

I Am informed that some Papers have been presented to the Royal Society of late, concerning the Description of Curves, in a manner that has a near affinity to that which I communicated to them of old, and have carried farther since; and that it would not be unseasonable, nor unacceptable, if I should send an Account of what I have done further on that Subject since the Year 1719. The Author of those Papers taught Mathematicks here privately for some Years, and sometime ago (*viz.* in 1727.) mentioned to me some Theorems he had on that Subject; which, at the same time, I shewed him in my Papers. Some time before that, he shewed me a Theorem which coincided with one of those in my Book, tho' he seemed not to have observed that Coincidence; and indeed Methods of that kind, are often found coincident that do not appear such at first sight. I am unwilling to be the Occasion of discouraging any thing that is truly ingenious, and renounce any Pretensions of appropriating Subjects to my self; but, on the contrary, wish Justice may be done to every Person, or to any Performance in Proportion to it Merit; yet I find

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it is fit I should take Precautions, lest any one should take it in his head afterwards to say, I take things from him which I may have had long before him; and therefore shall send you an Abstract of what I have done in relation to this matter, since the Year 1719.

I have so much on this Subject by me, that I am at a loss what to send; but at present I shall only give you an Abstract of those Propositions, which I take to be more nearly related to those which this Author has offered to the Society from the Conversations I had with him. You know that in 1721, I printed several Sheets of a Supplement to my Book on the Description of Curve Lines, which I have never yet published, having been engaged for the most part in Business of a different nature, and in Pursuits on other Subjects since that time. I shall first give you an Abstract of that Supplement, as far as it was then printed, and shall subjoin to this, an Account of some Theorems I added to it the following Year, *viz.* in 1722. I was led into those new Theorems by Mr. *Robert Sympfson's* giving me at that time a Hint of the ingenious Paper, which has been since published in the Philosophical Transactions. I had tried in the Year 1719, what could be done by the Rotation of Angles on more than two Poles; and had observed, that if the Intersections of the Legs of the Angles were carried over Right Lines, as in Sir *Isaac Newton's* Description, the Dimensions of the Curve were not raised by this Increase of the Number of Poles, Angles, and Right Lines; and therefore neglected

lected this at that time, as of no use to me ; confining my self to two Poles only, and varying the Motions of the Angles as you find them in my Book. I found this by inquiring in how many Points the Locus could cut a Right Line drawn in its Plane, and found, by a Method I often use in my Book, that it could meet it in two Points only.

Having found then, that three or more Poles, were of no more Service than two, while the Intersections were carried over fixed Right Lines ; I thought it needless to prosecute that Matter then, since by increasing the Number of Poles, my Descriptions would become more complex without any Advantage. But in *June* or *July*, 1722, upon the Hint I got from Mr. *Sympson* of Mr. *Pappus's* Porisms, I saw that what he has there ingeniously demonstrated, might be considered as a Case of the above-mentioned Description of a Conick Section, by the Rotation of any Number of Angles about as many Poles ; the Intersections of their Legs, in the mean time, being carried over fixed Right Lines, excepting that of two of them which describes the Locus. For by substituting Right Lines in place of the Angles, in certain Situations of the Poles and of the fixed Right Lines, the Locus becomes a Right Line ; as for Example, in the Case of three Poles, when these three are in one Right Line, in which Case the Locus is a Right Line, which is a Case of the Porism.

'Twas this led me to consider this Subject anew ; and first I demonstrated the Locus to be a Conick Section algebraically ; and found Theorems for

drawing Tangents to it, and determining its Asymptotes. I also drew from it at that time a Method of describing a Conick Section through five given Points *. This encouraged me to substitute Curves for the Right Lines, to see if by this Method I could be enabled to carry on my Theorems about the Descriptions of Lines through given Points to the higher Orders of Lines. Some of the Theorems I found at that time, I now send you. In *November 1722*, looking into Sir *Isaac's Principia*, I saw that the Description of the Conick Section by three Right Lines, moving as above, about three Poles, could be immediately drawn from his 20th Lemma, which itself is a Case of this Description. This gradually led me to seek Geometrical Demonstrations for the whole, as far as it related to the Conick Sections. I send you some Leaves of this Paper dated at *Nancy, November 1722*. Since that time, I have not added much to this Subject, but what relates to the drawing Tangents, determining the Asymptotes, and the *Puncta Duplica*, or *Multiplia* of these Curves. I considered it the less, that I did not find it more advantageous in any respects, than the Method I had considered in my Book, or more general.

In 1727 I added to a Chapter in my Algebra, which is very publick in this Place, an Algebraick Demonstration of the Locus, when three Poles are employed; and the Method of describing a Conick

* The Paper on this Subject I have, is dated July 31, 1722, at Sea, being then in my way to London, going for Cambray.

Section through five given Points, subjoining at the same time, that if more Poles are employed, and Angles or Right Lines, the Locus was still a Conick Section; which I thought was a remarkable Property of the Conick Sections not observed before.

These Things I intended to put in order, and publish in the Supplement to my Book, a Part of which has been printed since the Year 1721. I have in my view also to give several other Things in that Supplement; two of which, I shall only just mention at present, because I believe they are foreign to the present Affair. I subjoin a Problem determining the Figure of a Fluid, whose Parts are supposed to be attracted to two or more Centers; and a Solution of a general Problem about the Collision of Bodies.

The Author of the Papers given in to the Royal Society, will not refuse that I shewed him the Theorems, I now send you, in 1727. He owned it last Summer at least: I am to publish these very soon. Whether he has carried the Subject farther, I leave to the Judgment of the Gentlemen to whom they were referred. As to the Demonstrations, it would take some time to put them in a proper Form to be published. You who have so nice a Taste of Demonstrations, will easily allow, that it ought not to be done in a hurry. I could send those that are Algebraick easily; but do not care to send those that are Geometrical, till I have leisure. I could not have been called to this in a worse Season of the Year than now, when I begin my Classes, and have few

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few Minutes in the Day my own. I ought to make an Apology for this long Letter; but thought you was the Person of my Acquaintance most proper to send this to. I am, with great Esteem,

S I R,

Your most obedient

most humble Servant,

Colin Mac Laurin.

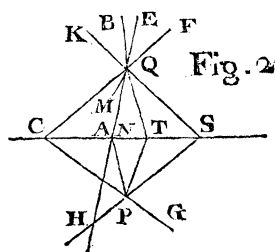
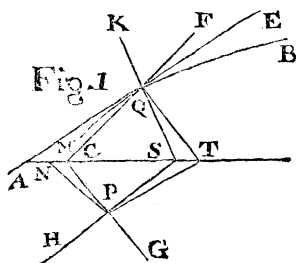
An Abstract of what has been printed since the Year 1721, as a Supplement to a Treatise concerning the Description of Curve Lines published in 1719, and of what the Author proposes to add to that Supplement.

I. IN the first Part of the Supplement, there is a general Demonstration given of the Theorem, that if two Lines of the Orders or Dimensions, express'd by the Numbers m and n , be described in the same Plane, the greatest Number of Points in which these Lines can intersect each other, will be mn , or the Product of the Numbers which express the Dimensions of the Lines, or the Orders to which they belong.

II. In the next Part, Theorems are given for drawing Tangents to all the Curves that were described in that Treatise by the Motions of Angles upon

upon given Lines. Their Asymptotes are also determined by more simple Constructions than those which are subjoined to their Descriptions in that Treatise. Of these we shall give one Instance here.

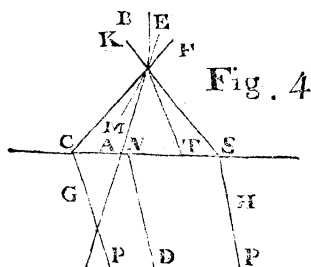
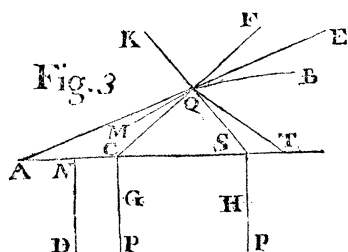
Suppose the invariable Angles (*Fig. 1* and *2*) FCG , KSH , to revolve about the fixed Points or



Poles, C and S . Suppose the Intersection of the two Sides CF , SK , to be carried over the Curve BQM , whose Tangent at the Point Q is supposed to be the Right Line AE ; and let it be required to draw a Tangent at P to the Curve Line described by P the Intersection of the other two Sides CG and SH .

Construction. Draw QT constituting the Angle SQL , equal to CQA , on the opposite Side of SQ , that QA is from CQ ; and let QT meet CS (produced if necessary) in T . Join PT , and constitute the Angle CPN equal to SPT , on the opposite Side of CP , that PT is from SP , and the Right Line PN shall be a Tangent at P , to the Curve described by the Motion of P , which is always supposed to be the Intersection of CG and SH .

The Asymptotes of the Curve, described by P, are determined thus. Find, as in the abovementioned Treatise, when these Sides become parallel, whose Interfection is supposed to trace the Curve ; which always happens when the Angle CQS becomes equal to the Supplement of the Sum of the invariable Angles FCG , KSH , to four Right ones, because the Angle CPS then vanishes. Suppose (in *Fig. 3* and 4,) that when this happens, the Interfection of the Sides CF , SK is found in Q .

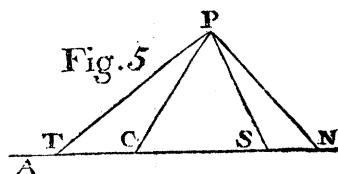


Constitute the Angle SQT equal to CQA , as before, and let QT meet CS in T . Take CN equal to ST , the opposite Way from C that ST lies from S . Through N draw DN parallel to CG or SH , which are now parallel to each other, and DN shall be an Asymptote of the Curve described by the Motion of P .

If in place of a Curve Line BQM , a fixed Right Line AE be substituted, then the Point P will describe a Conick Section, whose Tangents and Asymptotes are determined by these Constructions. In this Supplement, it is afterwards shewn how to draw the Tangents and Asymptotes of all the Curves
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which are described in the above-mentioned Treatise by more Angles and Lines.

III. The same Method is afterwards applied for to draw Tangents to Lines described by other Motions than those which are considered in that Treatise ; of which the following is an Instance. Suppose that the Lines CP and SP revolve about the Poles C and S , so that the Angle ACP bears always the same invariable Proportion to ASP , suppose that of m to n . In the Line CS , take



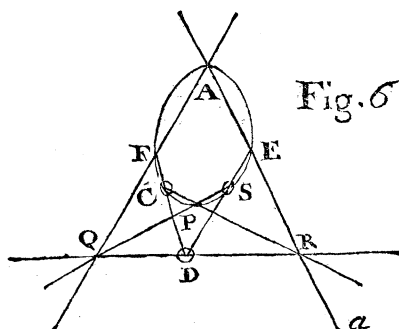
the Point T , so that ST may be to CT in that same Proportion of m to n ; and this Point T will be an invariable Point, since CT is to CS , as $m - n$ to n . Draw TP , and constitute the Angle SPN , equal to $CP T$, so that PN and PT , may lie contrary ways from SP and CP , and PN shall be a Tangent of the Curve described by the Motion of the Point P . Several other Theorems of this kind are subjoined here.

IV. After these, Lines or Angles are supposed to revolve about three or more Poles, and the Dimensions of the Curves with their Tangents and Asymptotes are determined. Suppose in the first Place,

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that the three Poles are C, S and D, and that Lines or Rulers CR, SQ, QDR, revolve about these Poles. The Line which revolves about D, serves only to guide the Motion of the other two, so that its Interfection

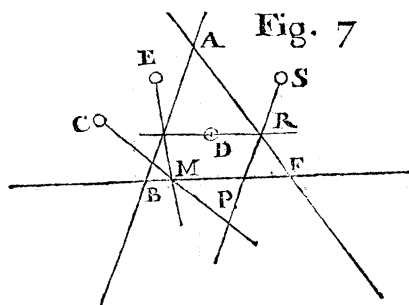


with each of them being carried over a fixed Right Line, their Interfection with each other describes the Locus, which is shewn to be a Conick Section. The Interfection of QDR with SQ, is supposed to be carried over the fixed Right Line AF; the Interfection of the same QDR with CR, is supposed to be carried over the fixed Right Line AE; and in the mean time, the Interfection of the Right Lines SQ, CR, that revolve about the Poles S and C, describes a Conick Section.

This Conick Section passes through the Poles C and S; and if you produce DC and DS, till they meet with AQ and HR in F and E, it will also pass through F and E: It also passes always through A the Interfection of the fixed Lines QF and ER; from which this easy Method follows for drawing a Conick Section through five given Points. Suppose that these five given Points are A, F, C, S and E: Join four of them by the Lines AF, FC, AE, ES, and produce two of these FC, ES, till they meet, and by their Interfection give the Point D. Suppose infinite Right Lines revolve about this Point

Point **D**, and the Points **C** and **S**, two of those that were given, and let the Intersections of the Line revolving about **D**, with those that revolve about **C** and **S**, be carried over the given Right Lines **A E**, **C F**; and the Intersection of those that revolve about **C** and **S** with each other, will, in the mean Time, describe a Conick Section, that shall pass through the five given Points **A**, **F**, **C**, **S** and **E**.

It is then shewn, that when **C**, **S** and **D** are taken in the same Right Line, the Point **P** describes a Right



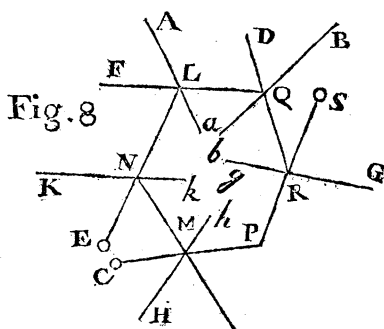
Line ; as also when **C**, **S** and **A** are in the same Right Line ; which also follows from what is demonstrated in that very ingenious Paper concerning *Pappus's Porisms*, communicated by Mr. *Sympson*, Professor of Mathematicks at *Glasgow* published in the *Phil. Transf.* No. 377.

In the next Place it is shewn, that if four Right Lines revolve about four Poles **C**, **S**, **D** and **E**, and those that revolve about **D** and **E**, serve only to guide those that revolve about **C** and **S** ; so that **Q** and **R**, the Intersections of that which revolves about **D**, with those that revolve about **E** and **S**, be carried over the fixed Lines **A B** and **A F** ; and **M** the

Intersection of that which revolves about E with that which revolves about C, be carried over a third fixed Line BF, then the Intersection P of those that revolve about C and S, will, in the mean time, describe a Conick Section, and not a Curve of a higher Order. The Conick Section degenerates into Right Lines, when CP and SP coincide at the same time with the Line CS, that joins the Poles C and S, as in the preceding Description; which coincides again with what is demonstrated in the abovementioned ingenious Paper.

After this it is shewn generally, that tho' the Poles and Lines revolving about them be increased to any Number, and the fixed Lines over which such Intersections, as we described in the two last Cases, are supposed to be carried, be equally increased, the Locus of the Point P will never be higher than a Conick Section: That is, let a Polygon of any number of Sides have all its Angles, one only excepted, carried over fixed Right Lines, and let each of its Sides produced, pass through a given Point or Pole, and that one Angle which we excepted, will either describe a streight Line, or Conick Section.

Thus if a hexagonal Figure LQ R P M N, have all its Angles excepting P carried re-



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spectively over the fixed Right Lines Aa , Bb , Gg , Hh , Kk , the Point P in the mean time will describe a Conick Section, or a Right Line. The Locus of P is a Right Line when CP and SP coincide together with the Line CS . All these things are demonstrated geometrically.

V. After this, Angles are substituted in place of Right Lines revolving about these Poles; and it is still demonstrated geometrically, that the Locus of P is a Conick Section or Right Line.

Suppose that there are four Poles C , S , D and E , about which the invariable Angles PCQ , PSR , $RD M$, MEQ revolve; and that Q , M and R , the Intersections of the Legs CQ and EQ , of EM and DM , and of DR and SR , are carried over the fixed Right Lines Aa , Bb , and Gg respectively, then the Locus of P is a Conick Section, when CP and SP do not coincide at once with the Line CS , but is a Right Line when CP and SP coincide at the same time with CS , and never a Curve of a higher Order.

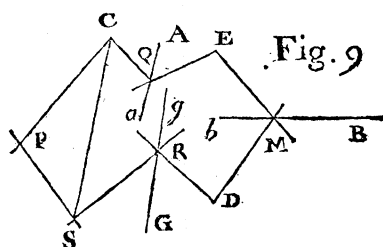


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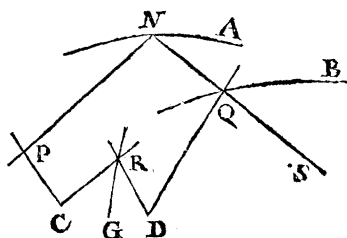
VI. Having demonstrated this which seems a remarkable Property of the Conick Sections or Lines of the Second Order; I proceed to substitute Curve Lines in place of Right Lines in these Descriptions, (as I always do in the Treatise concerning the Description

scription of Lines) and to determine the Dimensions of the Locus of P, and to shew how to draw Tangents to it to determine its Asymptotes, and other Properties of it. I had observed in 1719, that by increasing the Number of Poles and Angles beyond two, the Dimensions of the Locus of P, did not rise above those of the Lines of the Second Order, while the Intersections moved on Right Lines; and therefore I did not think it of use to me then to take more Poles than two, since by taking more, the Descriptions became more complex without any Advantage. When the Intersections are carried over Curve Lines, the Dimensions of the Locus of P rise higher, but the Curves described, have *Puncta Duplicia*, or *Multiplicia*, as well as when two Poles only are assumed; and therefore this Speculation is more curious than useful. However, I shall subjoin some of the Theorems that I found on this Subject concerning the Dimensions of the Locus of P, and the drawing Tangents to it.

1. If in *Fig. 6.* you suppose Q and R to be carried over Curve Lines of the Dimensions m and n respectively, then the Point P may describe a Locus of $2\ m\ n$ Dimensions.

2. If in *Fig. 8.* you suppose L, Q, R, M, N, to be carried over Curve Lines of the Dimensions m, n, r, s, t , respectively, the Locus of P may arise to $2\ m\ n\ r\ s\ t$ Dimensions, but no higher; and if in place of Lines revolving about the Poles, you use invariable Angles, the Dimensions of the Locus of P will rise no higher.

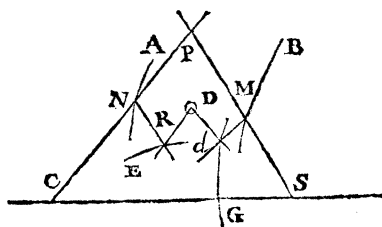
3. I then assumed three Poles C, D and S, and supposed one of the Angles S N L, to have its angular Point N carried over the Curve A N, while the Leg N Q passes always through S, as



in the Description in the Treatise of the General Description of Curve Lines, while the Angles Q D R, R C P, revolve about the Poles D and C : I suppose also the Intersections Q and R to be carried over the Curve Lines B Q, G R, and that the Dimensions of the curve Lines A N, B Q, G R, are m, n, r , respectively ; and find that the Locus of P may be of $3 m n r$ Dimensions ; but that the Point C is such, that the Curve passes through it as often as there are Units in $2 n m r$.

4. If any number of Poles are assumed, so as to have Angles revolving about them, as about C and D in the last Article, and the Intersections are carried over other Curves, the Dimensions of the Locus of P will be equal to the triple Product of the Number of Dimensions of all the Curves employed in the Description.

5. If the invariable Angles P N R, P M Q, move so that while the Sides P N, P M, pass always through the Poles C and S, the angular



Points

and the Dimensions of the Locus of P when highest, shall be equal to the Product of the Numbers that express the Dimensions of the given Curves multiplied by Six. If more Poles, with the necessary Angles and Curves, are assumed betwixt C and D, as here D is assumed betwixt C and S, and the Motions be in other respects like to what they are in this Example ; then in order to find the Dimensions of the Locus of P when highest, raise the Number 2 to a Power whose Index is less than the Number of Poles by a Unit ; add 2 to this Power, and multiply the Sum by the Product of the Numbers that express the Dimensions of the Curves employed in the Description ; and this last Product shall shew the Dimensions of the Locus of P when highest.

I am able to continue these Theorems much farther : But it is not worth while, especially since I find that there is not any considerable Advantage obtained by increasing the number of Poles above the Method delivered in the abovementioned Treatise of the Description of Curve Lines. On the contrary, the Descriptions there given by means of two Poles, will produce a Locus of higher Dimensions by the same number of Curves and Angles, than these that require three or more Poles ; and are therefore preferable, unless perhaps in some particular Cases.

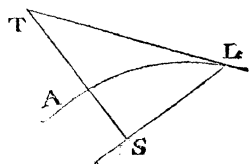
VII. However, I have also found how to draw Tangents to the Curves that arise in all these Descriptions ; of which I shall give one Instance where three Right Lines are supposed to revolve about three Poles, and two of their Intersections are supposed

I have also applied this Doctrine to the Description of Lines through given Points. But I suppose I have said enough at present on this Subject; and shall conclude, after observing that in the abovementioned Treatise, I have given an easy Theorem for calculating the Resistance of the Medium when a given Curve is described with a given centripetal Force in a resisting Medium, which I shall here repeat, because it has been misrepresented in a foreign Journal.

Let V express the centripetal Force with which the Body that is supposed to describe the Curve, is acted on in the Medium; let v express the centripetal Force with which the same Curve could be described in a Void; suppose $z = \frac{V}{v}$, and the Resistance shall be proportional to the Fluxion of z multiplied by the Fluxion of the Curve, supposing the Area described by a Ray, drawn from the Body to the Center of the Forces, to flow uniformly. Let this Theorem be compared with what the celebrated Mathematician mentioned by that Journalist has given on the same Subject, and it will easily appear what judgment is to be made of his Assertion; and since several Persons, and particularly the Gentleman mentioned above in this Paper, testify that I communicated to them this Theorem before any Thing was published on this Subject by the learned Mathematician he names, his Observation on this Occasion must appear the more groundless.

From this Theorem, I draw this very general Corollary, that if the Curve is such as could be described in a Void by a centripetal Force, varying according

cording to any Power of the Distance, then the Density of the Medium in any place, is reciprocally proportional to the Tangent of the Curve at that place, bounded at one Extremity by the Point of Contact, and, at the other, by its Intersection with a Perpendicular raised at the Center of the Forces to the Ray drawn from that Center to the Point of Contact. Let AL be the Curve described by a Force directed to the Point S ; let LT touch the Curve at L , and raise ST perpendicular to SL , meeting LT in T , and the Density in L shall be universally as LT , if the Resistance be supposed to observe the compound Proportion of the Density, and of the Square of the Velocity.



Besides what I have observed here, I propose to illustrate and improve several other Parts of the Treatise concerning the Description of Curve Lines in this Supplement.

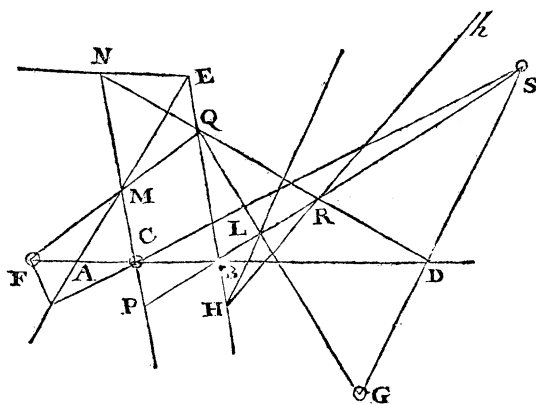
That Treatise requires these Additions and Illustrations the more, that tho' the whole almost was new, it was published in a hurry, when I was very young, before I had time to consider sufficiently which were the best ways of demonstrating the Theorems, or resolving the Problems, for which this Supplement I hope, will make some Apology.

N. B. *The following Paper, dated at Nancy, Novem. 27, 1722, is that which the Author mentions in his Letter.*

Si punctum D sumatur ubicunque in recta BF , sitque semper DG ad QG ut BD ad QF , eadem erit sectio conica quam d describet.

Sectio conica transit per C, P, B & a complendo parallelogrammum, $PSaQ$. Transit etiam per L ubi recta BG producta occurrit ipsi Pq , ut etiam per K ubi recta CD fecat datam PG . Unde pentagonum $PKCLB$ sectioni inscribitur. Et si detur quinque puncta C, K, P, B, L , per quæ ducenda est sectio conica vel circa pentagonum datum $CLBPK$ circumscribenda sit sectio conica producantur duo quævis latera CK, LB ad occursum suum D dein jungantur reliqua PL, PK & ducantur semper concursus rectarum Cd, Dr ; Bd, DR per rectas illas PL, PR , & concursus d sectionem describet.

P R O P. II.



Circa data puncta F, C, G, S , ut Polos moveantur, rectæ FQ, CN, GQ, SL & concursus rectarum
 FQ

FQ & CN , FQ & GQ , GQ & SL , puncta scil. M , Q , L , semper contingant rectas positione datas AE , BE , HL ; & concursus rectarum CN , SL describet sectionem conicam.

Occurrant rectæ AM , HR ipsi BQ in E & H . Jungantur CF & GS quæ sibi mutuo occurrunt in D , jungatur DQ quæ occurrat rectis CM , SL in N & R ; & si jungantur EN , & HR , erunt EN & HR , rectæ positione datæ per Lemma 1*. Quippe cum sint puncta F , C , D , in eadem recta Linea & concursus rectarum FM , CM & FQ , DQ percurrant datas rectas, concursus crurum CM , DQ , etiam continget datam. Et similiratione cum sint S , D , G in eadem recta concursus rectarum DQ , SL etiam continget datam.

Omissis igitur Polis F & G , invenienda est curva quam concursus rectarum CN , SL ; viz. P describet dum rectis CN , DN , SR revolventibus circa Polos C , D , S concursus rectarum CN , DN contingit datam EN & concursus rectarum SR , DN contingit datam HR , eam vero sectionem esse conicam ex Prop. præcedenti est manifestum.

N. B. The Papers referred to were published in a little Treatise entituled, Exercitatio Geometrica de descriptione Curvarum. London. 1733. 4to.